Part II. Power-Law and Maxwell Fluids

Employing the same kinematic wave theory used in the previous paper on Newtonian fluids (Part I), draw resonance in isothermal spinning is solved analytically for both power law and Maxwell fluids. The critical drawdown ratios at the onset of draw resonance are again found by equating twice the throughput wave residence time to the thread-line residence time yielding the same results reported by others through numerical calculations (Pearson and Shah, 1974; Fisher and Denn, 1976), thus confirming the premise that draw resonance is the result of the propagation of throughput waves.

SCOPE

As a natural evolution of research, stability analysis of draw resonance in isothermal spinning had been extended to power law and Maxwell fluids after researchers successfully solved the case for Newtonian fluids using perturbation and eigenfunction methods. Numerical calculations yielded results which explained draw resonance in terms of how it develops and behaves. However, there has been the remaining question: why, from a kinematics viewpoint, does draw resonance exist? The objective of this paper is thus to answer that question by solving the problem analytically.

CONCLUSIONS AND SIGNIFICANCE

The critical drawdown ratio required to cause the onset of draw resonance in isothermal spinning of power law and Maxwell fluids is obtained by equating twice the throughput wave residence time to the thread-line residence time, that is, the condition of steady oscillation of thread-line. The results are the same as those reported previously by numerical calculations (Pearson and Shah, 1974; Fisher and Denn, 1976). For power law fluids, the critical drawdown ratio is less than 19.744 if the power law exponent $n < 1$ and greater than 19.744 if $n > 1$. For Maxwell fluids, the critical drawdown ratio slowly increases with increasing fluid relaxation time until it reaches its maximum value of 39.43 with the corresponding maximum relaxation time of 0.01135. Beyond that relaxation time, the system is always stable (no draw resonance) as far as the drawdown ratio is attainable without breaking the thread-line.

An interesting point is that although the throughput wave velocities for power law and Maxwell fluids are functions of the drawdown ratio, the distance from the spinneret, and material functions (power law exponent and relaxation time, respectively) as compared to the case of Newtonian fluids where it is a function of the drawdown ratio only, the throughput wave residence time (the traveling time from the spinneret to the take-up) is the same for all Newtonian, power law, and Maxwell fluids. This is a confirmation of the fact that draw resonance is indeed a dynamic phenomenon, not a viscoelastic phenomenon. In other words, draw resonance is only altered, not caused, by fluid viscoelasticity.

We again confirm that draw resonance in (isothermal) spinning is the result of the propagation of throughput waves (disturbances) for all types of fluids.

Since the introductory remarks we made in the previous paper (Part I) are also pertinent here, we directly proceed into the derivation of results for power law and Maxwell fluids.

DRAW RESONANCE FOR POWER LAW FLUIDS SPINNING

The governing equations for isothermal spinning of power law fluids neglecting inertia, gravity, and surface forces on the thread-line and assuming a uniform velocity distribution across the thread-line cross section are as follows (see Ishihara and Kase, 1976, and Pearson and Shah, 1974):

Continuity equation:

$$\left(\frac{\partial A}{\partial t}\right)_x + \left[\frac{\partial (Av)}{\partial x}\right]_t = 0 \tag{1}$$

Momentum equation

$$\frac{\partial}{\partial x}\left[\eta A\left(\frac{\partial v}{\partial x}\right)_t\right] = 0 \tag{2}$$

Constitutive equation:

$$\eta = \mu\left(\frac{\partial v}{\partial x}\right)^{n-1} \tag{3}$$

The subscripts of x and t are included in the above to clearly distinguish each partial derivative.

Following the same procedure as for the Newtonian fluids (Hyun, 1978), we multiply $[\partial(Av)/\partial A]_x$ through (1) to derive the following throughput wave equation

$$\left[\frac{\partial (Av)}{\partial t}\right]_x + U\left[\frac{\partial (Av)}{\partial x}\right]_t = 0 \tag{4}$$

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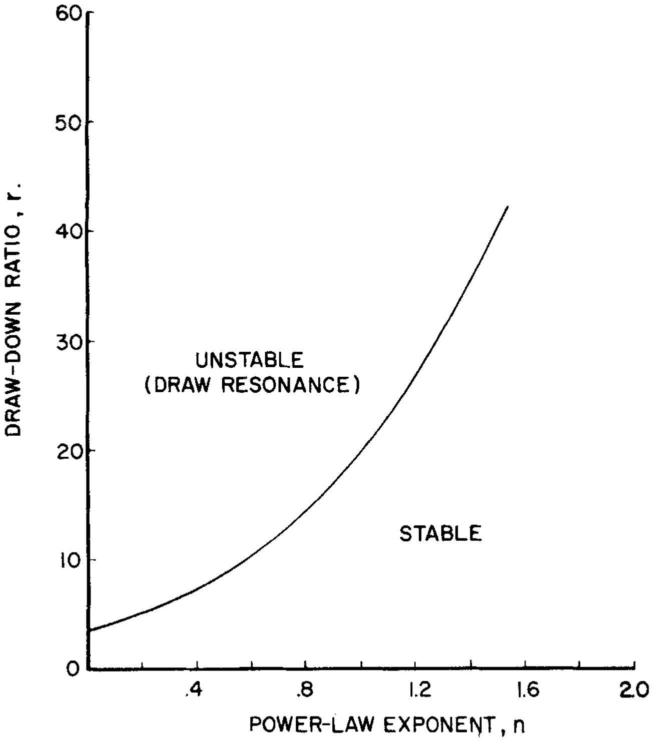

Fig. 1. Stability region for the spinning of power-law fluids.

with the throughput wave velocity

$$U = \left[\frac{\partial(Av)}{\partial A}\right]_x \tag{5}$$

which can be rewritten as

$$U = -\frac{A\left(\dfrac{\partial v}{\partial x}\right)_A}{\left(\dfrac{\partial A}{\partial x}\right)_{Av}} \tag{6}$$

The denominator is obtained from the steady state solution as follows:

$$A = A_o\left[1 + \frac{x}{L}\left(r^{\frac{n-1}{n}} - 1\right)\right]^{\frac{n}{1-n}}$$

$$\left(\frac{\partial A}{\partial x}\right)_{Av} = -A\frac{n}{n-1}\,\frac{r^{\frac{n-1}{n}} - 1}{L}\left[1 + \frac{x}{L}\left(r^{\frac{n-1}{n}} - 1\right)\right]^{-1} \tag{7}$$

As for the numerator of (6), we follow the same procedure as in Part I of Newtonian fluids, that is, a fictitious process where A is held constant. Then, since the thread-line force is always independent of x, we find $(\partial v/\partial x)_t$ also to be independent of x:

$$\therefore F = \mu A\left[\left(\frac{\partial v}{\partial x}\right)_t\right]^n \quad \text{from (2) and (3)}$$

Next, because of the constant A, that is, $(\partial A/\partial x)_t = 0$, we have $(\partial v/\partial x)_t = (\partial v/\partial x)_A$, and consequently $(\partial v/\partial x)_A$ is independent of x.

The simple integration of the above along with the boundary conditions of $v = v_o$ at $x = 0$ and $v = rv_o$ at $x = L$ yields

$$\left(\frac{\partial v}{\partial x}\right)_A = \frac{v_o(r-1)}{L} \tag{8}$$

at the onset of draw resonance. Substituting (7) and (8) into (6), we have

$$U = \frac{n-1}{n}\,\frac{v_o(r-1)}{r^{\frac{n-1}{n}} - 1}\left[1 + \frac{x}{L}\left(r^{\frac{n-1}{n}} - 1\right)\right] \tag{9}$$

which is an increasing function of x and is reduced to that of Newtonian case, that is,

$$U = \frac{v_o(r-1)}{\ln r}, \quad \text{as} \quad n \to 1.$$

Now we calculate the throughput wave residence time (traveling time from the spinneret to the take-up):

$$t_L = \int_o^L \frac{dx}{U}$$

$$= \frac{n}{n-1}\,\frac{r^{\frac{n-1}{n}} - 1}{v_o(r-1)}\,\frac{L}{r^{\frac{n-1}{n}} - 1}\int_o^L \frac{dx}{x + \dfrac{L}{r^{\frac{n-1}{n}} - 1}}$$

$$= \frac{L\ln r}{v_o(r-1)} \tag{10}$$

Note that the throughput wave residence time is independent of n and also identical to that of the Newtonian case (Hyun, 1978), although the throughput wave velocity isn't as shown in (9). From the steady state results of isothermal spinning of power law fluids (Hyun and Ballman, 1978), we have the following expression for the thread-line residence time:

$$\tau_L = \frac{L(n-1)(1 - r^{-1/n})}{v_o\left(r^{\frac{n-1}{n}} - 1\right)} \tag{11}$$

Then, by equating $2t_L$ to τ_L (that is, for a steady oscillation (draw resonance) of thread-line, two waves should be able to travel the whole spinning distance in one thread-line residence time), we can find critical drawdown ratios as a function of n.

Figure 1 shows the critical drawdown ratio plotted against n. At $n = 1$, the value is 19.744, the same as was found in Part I. The curve in Figure 1 is almost identical to what Pearson and Shah (1974) obtained by using an eigenvalue method.

DRAW RESONANCE OF MAXWELL FLUIDS SPINNING

Here we use an approximate model for isothermal spinning of convected Maxwell fluids by neglecting the isotropic pressure term, that is, dropping the coupling between two extra stress terms, in addition to the usual assumptions of negligible inertia, gravity, and surface forces, and a uniform velocity distribution across the thread-line cross section. The omission of the isotropic pressure term is adopted to make an analytical procedure possible and thus better understand the physics of the process. The errors introduced by this approximation are minimal as compared with the numerical calculation results as will be shown later.

Thus, the governing equations are now as follows:

Continuity equation:

$$\left(\frac{\partial A}{\partial t}\right)_x + \left[\frac{\partial (Av)}{\partial x}\right]_t = 0 \qquad (12)$$

Momentum equation:

$$\frac{\partial}{\partial x}(A\sigma) = 0 \qquad (13)$$

Constitutive equation:

$$\sigma + \lambda\left[v\left(\frac{\partial \sigma}{\partial x}\right)_t - 2\sigma\left(\frac{\partial v}{\partial x}\right)_t\right] = 2\mu\left(\frac{\partial v}{\partial x}\right)_t \qquad (14)$$

The subscripts x and t are included to distinguish each partial derivative clearly.

The expression for the throughput wave velocity is the same as for other types of fluids because the same continuity equation is involved:

$$U = -\frac{A\left(\frac{\partial v}{\partial x}\right)_A}{\left(\frac{\partial A}{\partial x}\right)_{Av}} \qquad (15)$$

From the steady state solutions of (12), (13), and (14), we obtain

$$\left(\frac{\partial A}{\partial x}\right)_{Av} = -\frac{A}{v}\left(\frac{\partial v}{\partial x}\right)_{Av} = -\frac{A}{k + \lambda v} \qquad (16)$$

where

$$k = \frac{2\mu Q}{F}$$

As for the numerator of (15), we follow the same procedure as before, that is, a fictitious process where A is held constant. Again, because of the independency of the thread-line force with respect to x, $(\partial v/\partial x)_t$ also becomes independent of x:

$$\therefore F = \frac{2\mu A(\partial v/\partial x)_t}{1 - 2\lambda(\partial v/\partial x)_t} \quad \text{from (2) and (3)}$$

Then, because of the constant A, $(\partial v/\partial x)_A$ also becomes independent of x, the integration of which along with the boundary conditions of $v = v_o$ at $x = 0$ and $v = rv_o$ at $x = L$ yields

$$\left(\frac{\partial v}{\partial x}\right)_A = \frac{v_o(r-1)}{L} \qquad (17)$$

at the onset of draw resonance. The substitution of (16) and (17) into (15) yields

$$U = \frac{v_o(r-1)}{L}(k + \lambda v) \qquad (18)$$

and thus the throughput wave residence time becomes

$$t_L = \int_o^L \frac{dx}{U} = \frac{L}{v_o(r-1)}\int_o^L \frac{dx}{k + \lambda v}$$

Since $dv/dx = v/(k + \lambda v)$, from the steady state solution

$$t_L = \frac{L \ln r}{v_o(r-1)} \qquad (19)$$

Note here that the throughput wave residence time is independent of λ and also identical to those of Newtonian and power law fluids cases, that is, (19) is the same as (10), although the throughput wave velocity is different for the three different types of fluids.

The thread-line residence time for steady isothermal spinning of Maxwell fluids is from Hyun and Ballman (1978):

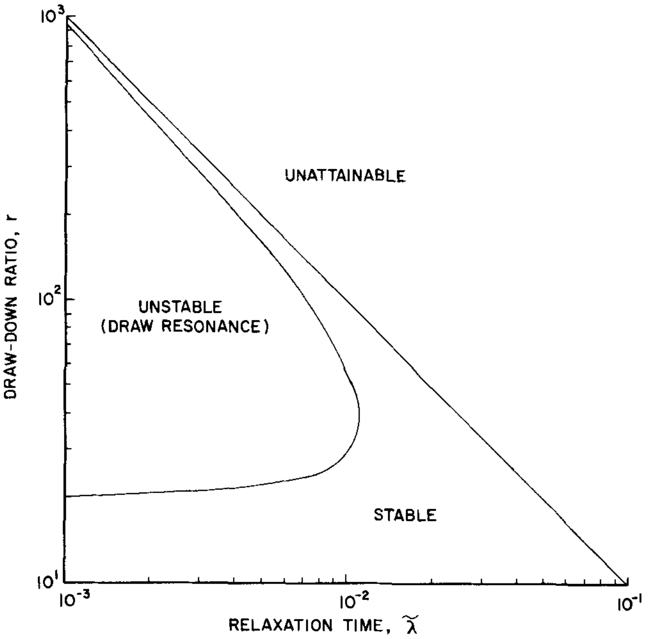

Fig. 2. Stability region for the spinning of Maxwell fluids.

$$\tau_L = \lambda \ln r + \frac{k}{v_o}\left(1 - \frac{1}{r}\right)$$

$$= \lambda \ln r + \frac{\left(1 - \frac{1}{r}\right)}{v_o \ln r}[L - \lambda v_o(r-1)] \qquad (20)$$

The condition for draw resonance, that is, $2t_L = \tau_L$, gives from (19) and (20)

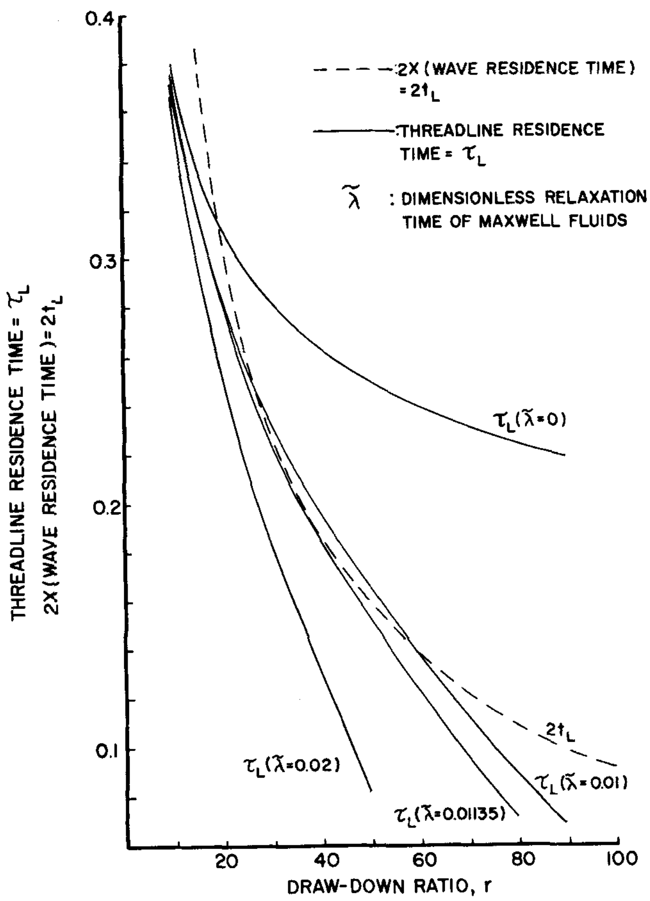

Fig. 3. Wave residence time and threadline residence time vs. draw-down ratio with varying relaxation time.

$$\tilde{\lambda} = \frac{\lambda v_o}{L} = \frac{2r(\ln r)^2 - (r-1)^2}{(r-1)[r(\ln r)^2 - (r-1)^2]} \quad (21)$$

as the relationship between the critical drawdown ratio and fluid relaxation time.

Figure 2 shows the curve of (21) together with the relationship between the maximum drawdown ratio and the relaxation time, which is obtained from (20):

$$k = \frac{2\mu Q}{F} = \frac{1}{\ln r}[L - \lambda v_o(r-1)] > 0$$

Thus

$$r_{\max} = 1 + \frac{1}{\tilde{\lambda}} \quad (22)$$

The plot in Figure 2 is almost identical to that obtained by Fisher and Denn (1976) through numerical calculations in their Figure 5 with $n = 1$. This proves two things: the assumption of zero isotropic pressure turns out to be a very good one with negligible error and greater simplicity of analytical solution, and draw resonance in isothermal spinning is indeed the result of the propagation of throughput waves along the thread-line.

The reason for the different regions in Figure 2 is explained by considering the magnitudes of $2t_L$ and τ_L as functions of r and λ as shown in Figure 3.

When $\tilde{\lambda} = 0$, that is, Newtonian fluid, the curve of $2t_L$ meets that of τ_L at a single point at $r = 19.744$. For $o < \tilde{\lambda} < \tilde{\lambda}_c = 0.01135$, the two curves meet each other at two points making two stability regions below and above the draw resonance region. When $\tilde{\lambda} = \tilde{\lambda}_c = 0.01135$, the two curves tangentially meet each other at a single point at $r = 39.43$, making the draw resonance point exist only in a theoretical sense. For $\tilde{\lambda} > \tilde{\lambda}_c$, the two curves don't meet each other so that there is no draw resonance region; drawing is limited only by the cohesive strength of the thread-line.

Two points deserve mention here. The drawdown ratio causing draw resonance at the critical relaxation time of 0.01135 is 39.43, almost twice that of Newtonian fluids. The significance of this is not clear at this time. The second point is that increasing relaxation time stabilizes the system; that is, the critical drawdown ratio increases, up to its critical value, beyond which it destabilizes the system in the sense of shrinking the attainable spinning region, although there is no draw resonance.

CONCLUDING REMARKS

In obtaining the same results of draw resonance for power law and Maxwell fluids as those previously reported (Pearson and Shah, 1974; Fisher and Denn, 1976) by adopting an analytical approach based on the kinematic wave theory, we established the fact that draw resonance in isothermal spinning is truly a phenomenon resulting from the propagation of throughput waves (disturbances) along the thread-line.

As for the effect of fluid relaxation time on the draw resonance in the case of real viscoelastic fluids, the picture is not clear at this time (for example, Weinberger et al., 1976), mainly because of difficulty in finding the best constitutive equation for a particular polymer.

For the fluids having constant relaxation times, draw resonance is stabilized by fluid elasticity as shown in Figure 2. However, for the fluids having strain rate dependent relaxation times, draw resonance could be either stabilized or destabilized by fluid elasticity, depending on the strain rate dependency of the relaxation time being small or large, respectively, as derived by Ide and White (1977). Experimental confirmation of this theory is not yet available.

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NOTATION

A = thread-line cross-sectional area
F = thread-line tension force
$\frac{k}{L}$ = dimensionless reciprocal thread-line force
L = distance from the spinneret to the take-up
n = power law exponent
r = drawdown ratio
t = time
t_L = throughput wave residence (traveling) time
U = throughput wave velocity
v = thread-line velocity
x = distance from the spinneret

Greek Letters

η = extensional viscosity of power law fluids
λ = relaxation time of Maxwell fluids
$\tilde{\lambda}$ = dimensionless relaxation time
μ = Newtonian extensional viscosity
σ = extra stress of thread-line
τ_L = thread-line residence time

Subscripts

o = conditions at the spinneret
L = conditions at the take-up
c = critical values

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